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SHAPE MATCHING AND MODULI SPACES

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Abstract. Shape matching represents a challenging problem in both information engineering and computer science, exhibiting not only a wide spectrum of multimedia applications, but also a deep relation with conformal geometry. After reviewing the theoretical foundations and the practical issues involved in this fashinating subject, we focus on two state-of-the-art approaches relying respectively on local features (landmark points) and on global properties (conformal parameterizations). Finally, we introduce the Teichmüller space $T_{g,n}$ of n-pointed Riemann surfaces of genus g into the realm of multimedia, showing that its beautiful geometry provides a natural unified framework for three-dimensional shape matching.

Admittedly, the purpose of the present contribution is not to prove any new mathematical result, but rather to suggest an intriguing connection between the area of three-dimensional shape matching and Teichmüller spaces of pointed Riemann surfaces. In order to address such an interdisciplinary topic in a hopefully transparent way, first of all we offer to the non-specialist reader a gentle introduction as well as an updated list of references to the technical literature on shape matching.

As incisively summarized in the survey paper [1], shape matching deals with transforming a shape and measuring the resemblance with another one via some similarity measure, thus providing an essential ingredient in shape retrieval and registration. By the way, there seems to be no universal and standard definition of what a shape is (see [1] and [2]): [1] goes even back to Plato's Meno, where Socrates claims in "terms employed in geometrical problems" that "figure is limit of solid" ([3]), while [2] relies on Kendall's definition of shape as "all the geometrical information that remains when location, scale, and rotational effects (Euclidean transformations) are filtered out from an object" ([4]).

The authors of [2] also emphasize the progressive shift of interest in the last decade from 2D (where shapes are just silhouettes) to 3D (where shapes are embedded surfaces), motivated by spectacular advances in the field of three-dimensional computer graphics. Unluckily, as remarked in [5], most 2D methods do not generalize directly to 3D model matching, thus inducing a flurry of recent research to focus on the specific problem of 3D shape retrieval. It is worth stressing that, unlike text documents, 3D models do not allow the simplest form of searching by keyword and it is now a rather

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common belief (see for instance [6]) that an efficient 3D shape retrieval system should take into account the principles of the human visual system as disclosed by cognitive neurosciences.

Before turning to the description of a couple of current approaches moving along these lines, we wish to mention at least another challenging application of three-dimensional shape matching. The so-called computational anatomy arises in medical imaging, in particular neuroimaging, and according to [7] it involves comparison of the shape of anatomic structures between two individuals, and development of a statistical theory which allows shape to be studied across populations.

A main component in this analysis, after obtaining the individual model representations for the subjects being studied, is the establishment of correspondence of anatomically homologous substructures between the subjects. For example, if we are interested in comparing shape differences between faces of two individuals in images, we would like to ensure that the coordinates of the left eye in one image correspond to the left eye in the other image. On a finer scale, we would like to ensure that the left corner of the left eye corresponds appropriately ([7]). More generally, following [8], we will define the landmarks of an object as the points of interest of the object that have important shape attributes. Examples of landmarks are corners, holes, protrusions, and high curvature points.

Landmark-based shape recognition is motivated by such a concept of dominant points. It uses landmarks as shape features to recognize objects in a scene or to establish correspondences between objects, by extending in an optimal way over the entire structure the correspondence at a finite subset. This last process is called landmark matching ([7]) and originates from the viewpoint of the human visual system, which suggests that some dominant points along an object contour are rich in information content and are sufficient to characterize the shape of the object (see [8] and the references therein). In particular, the paper [7] presents a methodology and algorithm for generating diffeomorphisms of the sphere onto itself, given the displacement of a finite set of template landmarks.

The restriction to a spherical domain is quite natural from the point of view of brain imaging, where a basic assumption is that the topology of the brain surface is the same as that of a crumpled sheet and, in particular, does not have any holes or self intersections ([9]). It then follows by Riemann Uniformization (see for instance [10], Theorem 4.4.1 (iii)) that there exists a conformal diffeomorphism of such a surface of genus zero onto a shere. More generally, every orientable compact embedded surface can be made into a Riemann surface with conformal structure (see for instance [11], Theorem IV.1.1).

As pointed out in [12] (see also references therein), the visual field is represented in the brain by mappings which are, at least approximately, conformal. Thus, to simulate the imaging properties of the human visual system conformal image mapping is a necessary technique. Moreover, beside its theoretical soundness, the application of conformal geometry to the 3D shape classification problem presents several practical advantages: according to [13], the conformal structures are independent of triangulation, insensitive to resolution, and robust to noises.

As far as the genus zero case is concerned, explicit conformal flattenings can be obtained by numerically solving the Laplace-Beltrami equation (the heart of this procedure consists in a finite element reduction to a system of linear equations, see [9] and [14]) or by deforming a homeomorphism in order to minimize the harmonic energy (both convergence of the algorithm and uniqueness of the solution are ensured by imposing further constraints, see [15] and [16]).

A natural generalization involves quasi-conformal mappings, which do not distort angles arbitrarily (as it is well-known, conformal mappings are angle-preserving): in particular, the least-squares conformal mappings (introduced in [17] via a least-squares approximation of the Cauchy-Riemann equations and in [16] through a discrete version of the harmonic energy minimization method) provide a natural solution to 3D nonrigid surface alignment and stitching ([16]), at least in genus zero.

In higher genus, a parameterization method based on Riemann surface structure has been recently developed in a series of papers by Shing-Tung Yau and collaborators (see in particular [18]). The idea is to segment the surface according to its conformal structure, parameterize the patching using a holomorphic one form, and finally glue them together via harmonic maps ([19]). The resulting surface subdivision and the parameterizations of the components turn out to be intrinsic and stable, and an explicit method for finding optimal global conformal parameterizations of arbitrary surfaces is described in [20].

As far as we know, this is by now the end of the story; however, we cannot resist to underline that our (perhaps tendentious) survey of the subject has presented three-dimensional shape matching as a classification problem for Riemann surfaces carrying (land)marked points: from our point of view, time has finally come to introduce moduli spaces into the picture!

Indeed, as remarked in [2], most shape representation schemes convert a shape into a feature vector, which is represented as a point of a feature space in a database. However, as stressed in [21] with David Mumford as a coauthor, there is no natural linear structure on the set of shapes, and therefore it is undesirable to simply map this set to a linear feature space.

In the same paper, entirely devoted to the 2D case, the infinite dimensional space of shapes (rigorously defined as simple closed curves in the plane up to translations and scaling) is equipped with the Weil-Petersson norm via Teichmüller theory. In particular, by taking the integral of the Weil-Petersson norm along a path as the lenght of this path, a geodesic is defined as the shortest path connecting the two shapes and its lenght yields a global metric on the space of shapes. Moreover, the shapes along that path represent a natural morphing of one into the other. By the way, this sounds as a wonderful but delicate fact: since the Weil-Petersson norm has negative curvature, it appears very likely to be true that there is a unique geodesic joining any two shapes, but (as the authors admit in footnote 9 of the journal version [22]) because the space is infinite dimensional this claim requires proof and this aspect of the metric does not seem to have been discussed in the literature.

What is really amazing is that in the harder 3D case this embarassing point magically disappears! Indeed, following [23], let us fix nonnegative integers g and n such that $2g-2+n > 0$, a compact connected oriented reference surface S_g of genus g and a sequence of n distinct points $(x_1, \dots, x_n)$ on S_g. By definition, the Teichmüller space $T_{g,n}$ is the space of conformal structures on S_g up to isotopies that fix $\{x_1, \dots, x_n\}$ pointwise. It is, in a natural way, a complex manifold of dimension $3g - 3 + n$.

The idea of classifying 3D shapes according to their conformal structure goes back to [13], where the general principle is stated that in nature it is highly unlikely for different shapes to share the same conformal structure. Of course by Riemann Uniformization this fails in genus $g = 0$, but the choice of a suitable number $n >> 0$ of landmark points allows to address in a uniform way also such an exceptional case. Indeed, we believe that the moduli space $T_{g,n}$ can provide a solid mathematical framework to three-dimensional shape matching, by supporting a unified geometric theory of landmark matching and conformal parameterizations.

Furthermore, the nice properties of the Weil-Petersson metric on this space allow a fully satisfactory solution to the problem raised above in 2D: as proved by Scott Wolpert in [24], every pair of points is joined by a unique geodesic. More precisely, geodesics are uniquely lenght minimizing and thus the Weil-Petersson distance between points is measured along the unique geodesic connecting them ([24], Corollary 5.9).

All the above undoubtedly points towards a potentially fruitful interaction between two such apparently unrelated fields as moduli spaces and information technologies. We hope to see deeper into this in the next future.

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