GIORNALE

DI MATEMATICHE

AD USO DEGLI STUDENTI

DELLE UNIVERSITÀ ITALIANE

PUBBLICATO PER CURA DEL PROFESSORE

G. BATTAGLINI

Volume XXXI. – 1893.

NAPOLI
BENEDETTO PELLERANO EDITORE
LIBRERIA SCIENTIFICA E INDUSTRIALE
Via Gennaro Serra, 20.
1893.

From the BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY *October*, 1893.

THE MATHEMATICAL CONGRESS AT CHICAGO

INAUGURAL ADDRESS

Delivered at the general session of the congress of mathematics and astronomy at Chicago august 21, 1893

BY

Prof. FELIX KLEIN.

« The German Government has commissioned me to communicate to this Congress the assurances of its good-will, and to participate in your transactions. In this official capacity allow me to repeat here the invitation given already in the general session to visit at some convenient time the German university exhibit in the Liberal Arts Building.

I have also the honor to lay before you a considerable number of mathematical papers, which give collectively a fairly complete account of contemporaneous mathematical activity in Germany. Reserving for the mathematical section a detailed summary of these papers, I mention here only certain points of more general interest.

When we contemplate the development of mathematics in this nineteenth century, we find something similar to what has taken place in other sciences The famous investigators of the preceding period, Lagrange, Laplace, Gauss, were each great enough to embrace all branches of mathematics and its applications. In particular, astronomy and mathematics were in their time regarded as inseparable.

With the succeeding generation, however, the tendency to specialization manifested itself. Not unworthy are the names of its early representatives: Abel, Jacobi, Galois, and the great geometers from Poncelet on, and not inconsiderable are their individual achievements. But along with its rapidly growing development the sciences departed more and more from its original scope and purpose, threatening to sacrifice its earlier unity and to split into diverse branches. At the same time the attention bestowed upon it by the general scientific public diminished in equal proportion. It became almost the custom to regard modern mathematical speculation as something having no general interest or importance; and the proposal was even made that, at least for purposes of instruction, all results be formulated from the same standpoints as in the earlier deriod. Such conditions were unquestionably to be regretted.

This is a picture of the past. I wish on the present occasion to state and to emphasize the fact that in the last two decades a marked improvement from within has asserted itself in our science, with constantly increasing succes.

The matter has been found simpler than was at first believed. It appears, indeed, that the different branches of mathematics have actually developed not in diverging but in parallel directions; that it is possible to combine their results into certain general conceptions. Such a conception is that of the *function* — in particular that of the analytical function of the complex variable. Another conception of perhaps the same range is that of the *group*, which just now stands in the foreground of mathematical progress. Proceeding from the idea of groups, we learn more and more to co-ordinate and connect different mathematical sciences. Thus, for example, geometry and the theory of numbers, which long seemed to represent antagonistic tendencies, no longer form an antithesis, but have come in many ways to appear as different aspects of one and the same theory.

This unifying tendency, originally purely theoretical, must inevitably extend to the applications of mathematics in other sciences, and on the other hand is sustained and reinforced in the development and extension of these latter. I presume that specific examples of this interchange of influence between pure and applied mathematics may be not without interest for the members of this general session, and on this account have selected for brief preliminary mention two of the papers which I have later to present to the mathematical section.

The first of these papers (by Dr. Schönflies) presents a review of the progress of mathematical crystallography. Sohncke, about 1877, treated crystals as aggregates of congruent molecules of any shape whatever, regularly arranged in space. In 1884 Fedorow made further progress by admitting the hypothesis that the molecules might be in part inversely instead of directly congruent. In the light of our modern mathematical developments this problem is one of the theory of groups, and we have thus a convenient starting-point for the solution of the entire question. It is simply necessary to enumerate all discontinuous groups contained in

the so-called principal group of space-transformations. From this point of view Dr. Schönflies has treated the subject in a text book (1891), while in the present paper he discusses the details of the historical development.

In the second place, I will mention a paper vhich has more immediate interest for astronomers, namely, a *résumé* by Dr. Burkhardt of the relations between astronomical problems and the theory of linear differential equations. This paper deals with those new methods of computing perturbations that were brought out first in your country by Newcomb and Hill; in Europe, by Gyldén and others. Here the mathematician can be of use to the astronomer, since he is familiar with linear differential eqnations and his trained in the deduction of strict proofs; on the other hand, the professional matematician finds here much to be learned. Hill's researches involve indeed — a fact not yet sufficiently recognized — a distinct advance upon the current theory of linear differential equations. To be more precise, te interest centres in the representation of the integrals of a differential equation in the vicinity of an « essentially » singular point. Hill furnishes a practical solution of this problem by the aid of an instrument new to mathematical analysis, — the admissibility of which is, however, confirmed by subsequent writers, — the infinitely extended, but still convergent, determinant.

Speaking, as I do, under te influence of our Göttingen traditions, and dominated somewhat, perhaps, by the great name of Gauss, I may be pardoned if I characterize the tendency that has been outlined in these remarks as a *return to the general Gaussian programme*. A distinction between the present and the earlier period lies evidently in this: that what was formerly begun by a single master-minde we must now seek to accomplish by nnited efforts and coöperation. A movement in this direction was started in France some time since by the powerful influence of Poincaré. For similar purposes we founded in Germany a mathematical society, three years ago, and I greet the young society in New York and its Bulletin as being in harmony with our aspirations. But mathematicians must go still farther. They must form international unions, and I trust that this present congress at Chicago will be a step in that direction ».

The mathematical section of the Congress on Mathematics and Astronomy held in Chicago from August 21st to 26th was of the highest interest to all present, particularly on account of the active participation of Prof. Klein, of Göttingen. Only a brief outline of the proceedings can be presented here, but it is hoped that a full official report of the proceedings will ultimately be published.

Monday's session was devoted to preliminary addresses and to organization,

Prof. Klein referring in his introductory address to two of the special papers presented:

Gruppentheorie und Krystallographie, by Prof. Schönflies of Göttingen, and

Ueber einige mathematische Resultate neuerer astronomischer Untersuchungen, insbesondere über irreguläre Integrale linearer Differentialgleichungen, by Dr. Burkhardt of Göttingen.

The so-called « structure theory » of crystals deals essentially with a problem of the theory of groups, namely, with the enumeration of all discontinuous sub-groups which may be formed from the group of space-movements, combined also with reflection. Dr. Schönflies has already (1891) ma le a comprehensive presentation of the theory from this standpoint.

Dr. Burkhardt gives a critical account from a mathematical standpoint of the interesting advances made in the last twenty years by Hill. Gyldén, and others in the field of perturbation calculation. These authors having employed linear differential eqnations for defining their transcendental functions, their investigations stand in close relations with developments in the theory of these equations made in more recent years from the purely mathematical side.

At the second session (August 22) the following papers were presented:

Ueber die Theorie der algebraischen Invarianten, by Prof. Hilbert of Königsberg, giving an account of the results reached in the investigations on the finiteness of the form systems, in which by the introduction of fundamentally ne w methods he has passed far beyond the previous standpoint.

Zur Theorie der ganzzahligen algebraischen Gleichungen, by Prof. Weber of Göttingen, giving an elementary proof that there is an infinite number of equations of prime order and with integral coefficients, which in the domain of rationality have no « Affect, » — equations, that is, whose Galois group is the symmetrical group.

Ueber die arithmetisch-algebraischen Tendenzen Leopold Kroneckers, by Prof. Netto of Giessen, illustrating the tendencies followed by Kronecker, particularly in his later years, to reduce the entire field of matematics to the *relations between integral numbers*, and to consider only such mathematical operations legitimate as involve but a finite number of steps.

Ueber die Reduction der binären quadratischen Formen, by Prof. Hurwitz of Zürich, a derivation, by a particularly simple and elegant geometrical method, of known theorems on the reduction of the binary quadratic forms.

On Fifth-power Numbers, whose Sum is a Fifth Power, by Dr. Artemas Martin of the U. S. Coast and Geodetic Survey, giving special numerical cases.

On the Algebraic Solution of Equations, by Prof. Sawin of Evansville, Wisconsin.

Aeltere und neuere Untersuchungen über Systeme complexer Zahlen, by Prof. Study of Marburg. The author deals with those complex numbers for which the product of two units is a linear combination of the original units; he gives the historical development of the theory, inviting special attention to the relations of the considerations involved to Lie's theory of continuous transformation-groups.

On the Definitions of the Trigonometric Functions, by Prof. Macfarlane of the University of Texas.

The session of Wednesday opened with a paper on *Modern Graphical Devel lopments*, by President Eddy of the Rose Polytechnic Institute, reviewing the development of graphical methods since the time of Monge, with particular reference to M. Lévy's « La statique graphique et ses applications aux Constructions ».

Some Salient Points in the History of Non-Euclidean and Hyper-Spaces were presented by Prof. Halsted of the University of Texas, with an account of the work of an Italian priest, Saccheri, published in 1733, containing a « statement of propositions in Lobatscheffsky's Non-Euclidean Geometry, with their synthetic proof in pure geometric style ».

Prof. Study of Marburg then presented a paper on *Some Researches in Spherical Trigonometry*, dealing with the formulæ of the latter from the standpoints of the modern theory of functions and of the theory of groups, and exhibiting close relations to various other branches of mathematics: the theory of orthogonal substitutions, that of desmic surfaces of the fourth order, that of the theta-relations, etc. (See the author's detailed account in the Abhandlungen der sächsischen Akademie, 1893). This was followed by a paper

On Interpolation Formulae and their Relation to Infinite Series, by Prof. Echols of the University of Virginia; by a

Résumé de quelques résultats relatifs à la théorie des systèmes récurrentes de fonctions, by Prof. Pincherle of Bologna.

Sur une intégrale définie qui représente la fonction ζ (s) *de Riemann*, by Prof. Lerch of Prague.

Ueber Eigenschaften von ganzen Zahlen, die durch räumliche Anschauung erschlossen sind, by Dr. Minkowski of Bonn, giving a preliminary account of the present state of the geometrical investigations on parallel grates (Parallelgitter) in space of n dimensions, by means of which he has derived a series of remarkable results in the theory of numbers.

Ueber die nothwendigen und hinreichenden Bedingungen für die Entwickelbarkeit von Functionen einer reellen Variablen nach der Taylorschen Reihe, by Prof. Pringsheim of Munich. The author gives first a review of our present knowledge of criterions of convergence of infinite series, then derives the conditions named in the title.

Consecutive und coincidirende Elemente einer algebraischen Curve, by Prof. Nöther of Erlangen, explaining anew the methods he has employed in the investigation of singular points, and showing the simplicity with which the chief theorems may be expressed.

Thursday's programme included the following papers:

The Principles of the Elliptic and Hyperbolic Analysis, by Prof. Macfarlane of the University of Texas, in continuation of his earlier paper.

On Weierstrass's Systems of Abelian Integrals of the First and Second Kinds, by Prof. Bolza of the University of Chicago, a precise derivation of certain fundamental theorems, using the methods of Weierstrass.

Fortschritte in der Theorie der linearen Differentialgleichungen, by Dr. Heffter of Giessen, a review of the researches of Fuchs and other German writers during the past five years.

Automorphe Functionen und Zahlentheorie, by Dr. Fricke of Göttingen, giving an account of the relations to arithmetical developments of the numerous discontinuous groups occurring in the theory of automorphic functions, so far as such relations have been investigated by himself and others.

Zur Transformation fünften Grades der hyperelliptischen Functionen erster Ordnung, by Prof. Krause of Dresden.

Sur quelques propositions fondamentales de la théorie des fonctions elliptiques, by Prof. Hermite of Paris, giving a new derivation of the addition theorem of the elliptic functions for any assigned biquadratic form under the radical sign in the elliptic integral.

The following papers were included in the programme for Friday:

On Orthogonal Substitutions, by Prof. Taber of Clark University, giving the determination of all possible real orthogonal substitutions (and of imaginary orthogonal substitutions with two, three, four, or six variables), rationally in terms of the minimum number of parameters; the determination of all symmetric orthogonal substitutions of n variables rationally in terms of $\frac{1}{2}n(n-1)$ parameters; and certain generalizations of Stieltjes' theorem, with an exponential representation of orthogonal substitutions.

On a Quaternary Group of 2520 *Linear Substitutions*, by Prof. Maschke of the University of Chicago, giving an account of the complete form system of that quaternary group of 2520 substitutions occurring in line geometry.

On Simple Groups, by Prof. Cole of the University of Michigan, describing a new simple group of 504 substitutions of nine letters.

A Doubly infinite System of Simple Groups, by Prof. Moore of the University of Chicago. The theory of the elliptic modular functions gives a simple group (known since the time of Galois of $\frac{q(q^2-1)}{2}$ substitutions, where q is

any prime number greater than 2. By introducing substitutions wit Galois imaginaries, the author obtains an entire series of new simple groups with $\frac{q^n(q^{2n}-1)}{2}$ substitutions. These results hold also for $q=2$ if the denominator 2 be omitted. For $q=2$, $n=3$, we obtain the group discussed by Prof. Cole.

At the final session (August 26) the following papers were presented:

A Formulary an Introduction to Elliptic Functions, by Prof. Stringham of the University of California. The author develops a system of formulae for the reduction of the elliptic integral of the first kind to different normal forms with corresponding functional notations, distinguishing the different types by the particular Cayleyan transformations employed in deriving them. The earlier formulae of Jacobi and Abel are then derived as well as those of Weierstrass.

A Construction of Galois's Group of 660 Elements, by Mr. Joseph de Perrott of Clark University.

Tabellen von endlichen continuirlichen Transformationsgruppen, by Prof. Meyer of Clausthal. One of the most remarkable results of the theory of Lie is that for a given number of variables, for example in the plane, there is only a limited number of distinct types. While Lie, however, has developed these equations only in the abridged infinitesimal form, the present paper contains complete lists of these types in finite form.

Einige Sätze vom Schwerpunkt, and *Der pythagoräische Lehrsatz in mehrdimensionalen Räumen*, by Prof. Schlegel of Hagen.

Considérations générales sur la mesure de la simplicité dans les sciences mathématiques et applications à l'évaluation théorique de la simplicité des tracés géométriques, and *Règle des analogies dans le triangle et transformation continue*, by M. Émile Lemoine of Paris.

Sur l'équation des lignes géodesiques, by Prof. Weyr of Prague.

Nomographie: Sur les équations représentables par trois systèmes rectilignes de points isoplèthes, by M. d'Ocagne of Paris.

Note concerning Arithmetical Operations involving Large Numbers, by Rev. T. M. Pervouchine of Kasan.

Quelques formules relatives aux operations de polaire, by Prof. Capelli of Naples.

Sul moto di rotazione di un corpo rigido attorno ad un punto fisso, by Prof. Paladini of Pisa.

Concerning the Development of the Theory of Groups during the last Twenty Years, by Prof. Klein of Göttingen. From his Erlangen Programme of 1872 (recently translated in the BULLETIN) as a starting point, Prof. Klein discusses briefly the development of the theories both of continuous and of discontinuous groups, referring to his recent lecture course, Höhere Geometrie II (soon to be lithographed).

After a unanimous and enthusiastic vote of thanks to Prof. Klein for his efforts in promoting the interest and success of these meetings and for the courtesies extended by him as Prussian commissioner at the Exposition, and after congratulatory remarks by Prof. Story, the section adjourned.

On three afternoons during the sessions of the Congress, Prof. Klein gave highly interesting expositions of the very complete Exhibit of the German Universities at the Liberal Arts Building. Two afternoons were devoted mainly to the explanation and discussion of mathematical models and other appliances, of which an extensive collection had been arranged by Prof. Klein and Dyck. Many of the models were unfamiliar to those present, and the opportunity for their examination was highly appreciated.

The officers of the mathematical section were:

Honorary President, Prof. Klein of Göttingen.

President, Prof. Story of Clark University.

Vice-President, Prof. Moore of the University of Chicago.

Secretary, Prof. Tyler of the Massachusetts Institute of Technology.

Executive Committee, the officers and Prof. White of Northwestern University.

Besides the officers just named the following were among those attending the congress: Prof. Study of Marburg, Halsted and Macfarlane of the University of Texas, Eddy of the Rose Polytechnic Institute, Bolza and Maschke of the University of Chicago, Paladini of Milan, Oliver and Macmahon of Cornell University, Van Vleck of Wesleyan University, Van Velzer of the University of Wisconsin, Beman and Ziwet of the University of Michigan, Smith of the University of Missouri, Fine of Princeton, Waldo of De Pauw University Merriman of Lehigh University, Lond of Colorado College, Taber and Webster of Clark University, Ely of Vassar, and Messrs. Hulburt of Johns Hopkins University, Holgate of Northwestern University, and Blake of Columbia College.

[For most of the above details in regard to particular papers the writer is indebted to the courtesy of Prof. Klein].

H. W. Tyler.
